J Math Chem (2010) 48:128–136
DOI 10.1007/s10910-009-9642-1

ORIGINAL PAPER

# Exploiting numerical behaviors in SPH

**G. Di Blasi · E. Francomano · A. Tortorici · E. Toscano**

Received: 24 September 2009 / Accepted: 24 November 2009 / Published online: 10 January 2010


**Abstract** Smoothed Particle Hydrodynamics is a meshless particle method able to evaluate unknown field functions and relative differential operators. This evaluation is done by performing an integral representation based on a suitable smoothing kernel function which, in the discrete formulation, involves a set of particles scattered in the problem domain. Two fundamental aspects strongly characterizing the development of the method are the smoothing kernel function and the particle distribution. Their choice could lead to the so-called particle inconsistency problem causing a loose of accuracy in the approximation; several corrective strategies can be adopted to overcome this problem. This paper focuses on the numerical behaviors of SPH with respect to the consistency restoring problem and to the particle distribution choice, providing useful hints on how these two aspects affect the goodness of the approximation and moreover how they mutually influence themselves. A series of numerical studies are performed approximating 1D, 2D and 3D functions validating this idea.

**Keywords** Meshless particle method · Smoothed particle hydrodynamics method · Consistency restoring · Function approximation · Particle distribution

## 1 Introduction

In simulating physical phenomena, it is well known that grid methods cannot perform well when irregular or deformable domains are considered. Valid computational alternatives to grid methods in the treatment of a wide variety of phenomena are meshless

G. Di Blasi (✉) · E. Francomano · A. Tortorici · E. Toscano
Dipartimento di Ingegneria Informatica, Università degli Studi di Palermo, Palermo, Italy
e-mail: diblasi@dinfo.unipa.it

methods. They share common features such as the avoidance of the use of grids, but are different in functions approximation and computational processes [4, 15].

Smoothed Particle Hydrodynamics (SPH) [10, 14, 18, 19] is a meshless method and got popularity due to its ability to evaluate unknown field functions and relative differential operators by means of an integral representation based on a suitable interpolating function [12]. The integral representation is discretized by using a set of *particles* scattered in the problem domain; this makes the method intrinsically adaptive.

In simulating real problems, it is often necessary to control regions with an high localized field gradient or to better reproduce irregular geometries of the problem domain. In these cases a regular particle distribution cannot be considered. This occurrence can lead to a lack of consistency of the method so that modified formulations have to be adopted to improve the quality of the approximation and different corrective strategies can be performed [1–3, 5, 6, 13, 16].

In this paper the numerical behaviors of SPH with respect to the consistency restoring problem and to the particle distribution choice are investigated; in effect, although the restoring of the consistency often leads to a good accuracy of the result, a further improvement can be obtained by judiciously distributing particles on the problem domain in such a way that they reproduce the properties of the function to approximate. In particular the approach proposed in [16] is applied to restore the particle consistency in approximating functions and studies on *ad-hoc* particle distributions are carried out. Some attentively considered examples in 1D, 2D and 3D domains are presented, showing the effectiveness of the presented idea.

The paper is organized as follows: Section 2 introduces the reader to a brief overview of the basic principles of SPH method, focusing on a technique [16] for the consistency restoring and on the issue of gather/scatter approach [11, 14, 22]; in Section 3 numerical experiments are carried out on 1D, 2D and 3D functions. Finally, Section 4 closes the paper by proposing future works.

## 2 Smoothed particle hydrodynamics: a brief overview

In order to approximate a function $f(\vec{x})$ in a domain $\Omega \subseteq \mathbb{R}^d$, SPH method works initially involving a function $W$ employed to define the *kernel approximation* of $f$:

$$f^h(\vec{x}) = \int_\Omega f(\vec{y})\, W(\vec{x} - \vec{y}, h)\, d\vec{y}. \tag{1}$$

The function $W$ is called the *smoothing kernel function* depending on the spatial variables and on the *smoothing length* parameter $h$:

$$W(\vec{x} - \vec{y}, h) = \alpha_d K(R), \tag{2}$$

where $R = \|\vec{x} - \vec{y}\| / h$ and $\alpha_d$ is a dimension-dependent normalization constant. Many kinds of smoothing kernel functions have been shown in SPH literature: the most popular are the bell-shaped functions [9, 10, 14, 17, 18, 20]. The smoothing kernel function should be symmetric, normalized, monotonic, with compact support and should satisfy the *Dirac delta function condition*:

$$\lim_{h \to 0} W(s, h) = \delta(s),$$

where $\delta(s)$ is the Dirac delta function and $s = Rh$.

By introducing a number of points (or *particles*) arbitrarily distributed to cover the problem domain, the kernel approximation can be discretized. The compactness condition says that only a finite number of particles referred as *nearest neighboring particles* (NNP) have to be considered for a satisfactory approximation. Therefore, kernel approximation is discretized by summing the contribution over all the NNP [10, 14, 18, 19] obtaining the so called *particle approximation*. Thus, the particle approximation of a function $f$ is obtained by "averaging" function values $f\left(\vec{x}_j\right)$ involving all NNP of $\vec{x}$:

$$f^h(\vec{x}) \cong \sum_{j=1}^{N} f\left(\vec{x}_j\right) W\left(\vec{x} - \vec{x}_j, h\right) V_j, \tag{3}$$

where $V_j$ is the measure of the support domain surrounding the particle $\vec{x}_j$.

The smoothing length $h$ and the number of particles determine the resolution of the approximation, so a crucial task before performing any computation using the SPH method is the NNP spotting. Two interpretations of the concept of nearest neighboring particles can be given [11, 14, 22]; these interpretations lead to different meanings for the $h$ parameter in equation (3): in the first one, historically known as *gather* approach, $h$ is referred as the radius size of the smoothing kernel support of particle $\vec{x}$, that is the subdomain where $\vec{x}$ "can search" its NNP; in the second one, known as *scatter* approach, $h$ is referred as the radius size of the smoothing kernel support of particle $\vec{x}_j$, that is the area where $\vec{x}$ has to lie in order to consider $\vec{x}_j$ a NNP of $\vec{x}$ (see Fig. 1). Moreover, following the gather model, according to equation (3), $f^h(\vec{x})$ is obtained by sampling $f$ on all NNP and summing $f\left(\vec{x}_j\right)$ weighted by the value of $W\left(\vec{x} - \vec{x}_j, h\right)$ in $\vec{x}_j$ (with $W$ centered in $\vec{x}$). Conversely, following the scatter model, $f^h(\vec{x})$ is computed by sampling $f$ on all NNP and summing $f\left(\vec{x}_j\right)$ weighted by the value of $W\left(\vec{x} - \vec{x}_j, h\right)$ in $\vec{x}$ (with $W$ centered in $\vec{x}_j$). Finally, by using the gather approach the particle approximation can be viewed as an interpolation technique, while if the scatter point of view is considered then the particle approximation assumes the meaning of an evelope [22].

Note that if the smoothing length $h$ has the same value for all particles then, when SPH is used to evaluate the function values on the same given particles, the use of the scatter or gather models is pratically equivalent. While, if SPH is used to evaluate the function value on an unknown particle the scatter model is mandatory, for this reason in this paper the scatter model is used. Finally, note that in solving real problems, the particle density could vary drastically leading to unbalanced smoothing kernel functions when the gather approach is considered; for this reason, to overcome this kind of problem, the scatter method should be used [15].

The particle approximation of a function $f$ can be affected by some numerical problems [14] such as the *particle inconsistency* [3, 21] which can lead to low approximation accuracy. In effect, the normalization and symmetric conditions required for the kernel function could not ensure the consistency conditions in the discrete formulation, that is the SPH ability to exactly reproduce a polynomial up to the $k$-th order (in this case the method is said to have $k$-th order of consistency) [3, 14, 17]. The particle inconsistency originates from the discrepancy between the SPH kernel and particle approximations: boundary particles, irregular and not judiciously distributed particles and variable smoothing length can usually produce inconsistency in the particle approximation process [14], so that different strategies have to be developed to restore the particle consistency [1–3, 5–7, 13, 16].

An interesting approach to restore the particle consistency, based on Taylor series expansion of $f(\vec{y})$ around $\vec{x}$, can be adopted by working with $k$-th order of particle consistency for both interior and boundary regions [16]. For the sake of simplicity, let $d$ and $k$ be equals to 1 (it is easy [8, 16] to extend the methodology to the $k$-th order of consistency and to any dimension). The methodology approximates $f$ in $\Omega \subseteq \mathbb{R}$, with 1-st order of consistency, as follows:

$$f(y) = f(x) + (y - x) f'(x), \tag{4}$$

next, by multiplying for the vector $\left(W(x-y,h), W'(x-y,h)\right)$ both the terms of (4) and by integrating on $\Omega \subseteq \mathbb{R}$, the following $2 \times 2$ system is obtained:

$$\begin{cases} \int\limits_{\Omega} f(y)\, W dy = \int\limits_{\Omega} f(x)\, W dy + \int\limits_{\Omega} (y-x) f'(x)\, W dy \\ \int\limits_{\Omega} f(y)\, W' dy = \int\limits_{\Omega} f(x)\, W' dy + \int\limits_{\Omega} (y-x) f'(x)\, W' dy, \end{cases} \tag{5}$$

where the argument $(x - y, h)$ in $W$ and $W'$ is omitted for clarity. Now the system (5) can be discretized and expressed as follows:

$$\begin{pmatrix} \sum_{j=1}^{N} \Phi_j^0 & \sum_{j=1}^{N} \left(x_j - x\right) \Phi_j^0 \\ \sum_{j=1}^{N} \Phi_j^1 & \sum_{j=1}^{N} \left(x_j - x\right) \Phi_j^1 \end{pmatrix} \cdot \begin{pmatrix} f(x) \\ f'(x) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N} f\left(x_j\right) \Phi_j^0 \\ \sum_{j=1}^{N} f\left(x_j\right) \Phi_j^1 \end{pmatrix}, \tag{6}$$

with $\Phi_j^0 = W\left(x - x_j, h\right) V_j$ and $\Phi_j^1 = W'\left(x - x_j, h\right) V_j$. By solving the linear system (6), function values and its derivatives are obtained with the 1-st order of consistency kept on all over problem domain and no modifications on the smoothing kernel function are required.

## 3 Numerical experiments

In this section the numerical behaviors of SPH with different consistency restoring approaches [16, 23] and different particle distributions are exploited by approximating

**Table 1** Summary of the numerical experiments

| Approximating function | Interval | Starting particles | Approximating particles |
|---|---|---|---|
| $f(x) = \cos(x)$ | $[0, 2\pi]$ | 50 | 500 |
| $f(x) = \cos(x)\,e^{-x/2\pi}$ | $[0, 48\pi]$ | 50 | 500 |
| $f(x) = \cos(x)\,e^{-\lvert x\rvert/2\pi}$ | $[-48\pi, 48\pi]$ | 100 | 1000 |
| $f(x, y) = \cos(x + y)$ | $[0, 2\pi]^2$ | $20^2$ | $50^2$ |
| $f(x, y, t) = \cos(x + y + t)$ | $[0, 2\pi]^3$ | $20^3$ | $50^3$ |

**Table 2** Root-mean-square errors obtained for the 1D functions

| Approximating function | Algorithm | Root-mean-square error | | |
|---|---|---|---|---|
| | | Even | LIU | S-LIU |
| $f(x) = \cos(x)$ | No consistency | 0.0610 | 0.0947 | 0.0947 |
| | 0-th order consistency | 0.0085 | 0.0418 | 0.0167 |
| | 1-st order consistency | 0.0021 | 0.0399 | 0.0123 |
| $f(x) = \cos(x)\,e^{-x/2\pi}$ | No consistency | 0.6117 | 0.4183 | 1.2947 |
| | 0-th order consistency | 0.4693 | 0.0515 | 1.2921 |
| | 1-st order consistency | 0.2622 | 0.0282 | 1.2647 |
| $f(x) = \cos(x)\,e^{-\lvert x\rvert/2\pi}$ | No consistency | 0.9639 | 1.2883 | 0.2658 |
| | 0-th order consistency | 0.9618 | 1.2746 | 0.0527 |
| | 1-st order consistency | 0.9617 | 1.2715 | 0.0289 |

functions in 1D, 2D and 3D domains. Computational results obtained are compared with the analytic functions profiles.

The well-known Gaussian function is chosen as smoothing kernel function:

$$W(\vec{x} - \vec{y}, h) = \alpha_d e^{-R^2}, \tag{7}$$

where $\alpha_d$ equals $1/\pi^{1/2}h$, $1/\pi h^2$, $1/\pi^{3/2}h^3$ respectively in one, two and three dimensions.

Table 1 reports the approximating functions, the intervals of approximation, the number of known particles and the number of approximated function values. For each function in Table 1, different particle distributions have been tested. In the 1D test cases, three distributions have been considered: the even distribution, the distribution proposed in [16] (from now on named as LIU) and a symmetric distribution (from now on named as S-LIU). These distributions have been judiciously selected, in fact the first one reproduces the regularity properties of the trigonometric functions, while the second and third distributions reproduce the irregular behavior of the second and third functions in Table 1. In the 2D and 3D test cases, only the even and LIU distributions have been taken into account, where the 2D-LIU and 3D-LIU distributions [8] are obtained by generalizing the idea proposed in [16] for the 1D case. In order to

**Table 3** Root-mean-square errors obtained for the 2D and 3D functions

| Approximating function | Algorithm | Root-mean-square error | |
|---|---|---|---|
| | | Even | LIU |
| $f(x, y) = \cos(x + y)$ | No consistency | 0.0559 | 0.3961 |
| | 0-th order consistency | 0.0303 | 0.0737 |
| | 1-st order consistency | 0.0262 | 0.0409 |
| $f(x, y, t) = \cos(x + y + t)$ | No consistency | 0.1022 | 0.2995 |
| | 0-th order consistency | 0.0631 | 0.1291 |
| | 1-st order consistency | 0.0568 | 0.1167 |

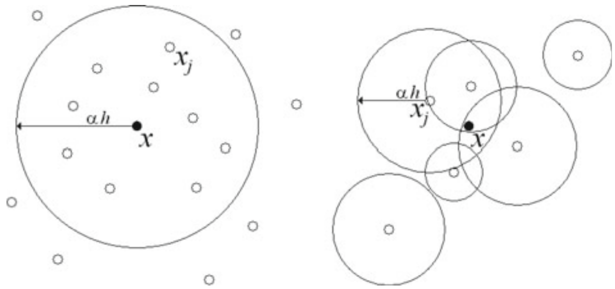


**Fig. 1** The smoothing kernels and the NNP for particle $\vec{x}$; on the left the gather model, on the right the scatter model

compare the accuracies of the obtained approximations the root-mean-square errors have been calculated and presented in Tables 2 and 3. Figure 2 shows the analytic profile of the third function, while Fig. 3 reports the approximation obtained without any consistency restoring and Fig. 4 presents the result obtained by restoring the 1-st order of consistency.

Obtained results show a general improvement of the approximation accuracy when the 1-st order of consistency is applied with respect to the 0-th order and the SPH without any consistency restoring, however a crucial role is played by the choiced particle distribution. For example, for each 1D test case, even if the restored consistency (0-th or 1-st order) improves the quality of the result, when it is used on a not-optimal distribution it is not able to gain orders of magnitude with respect to the SPH without any consistency restoring, while when the correct distribution is used it always gains one order of magnitude. The 2D and 3D test cases show similar behaviors to the 1D test cases providing an halved error when the right, i.e. even, distribution is used. Finally, the results in Tables 2 and 3 validate the idea that a good improvement of approximation quality can be reached by judiciously mixing a suitable technique of particle consistency restoring with an *ad-hoc* particle distribution able to mimic the regularity/irregularity properties of the function to approximate.

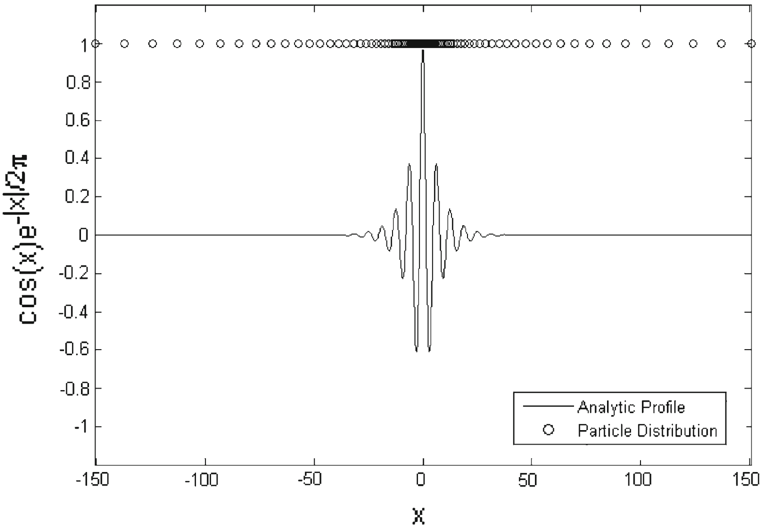


**Fig. 2** The analytic profile of the function $f(x) = \cos(x)\, e^{-|x|/2\pi}$ with the S-LIU particle distribution

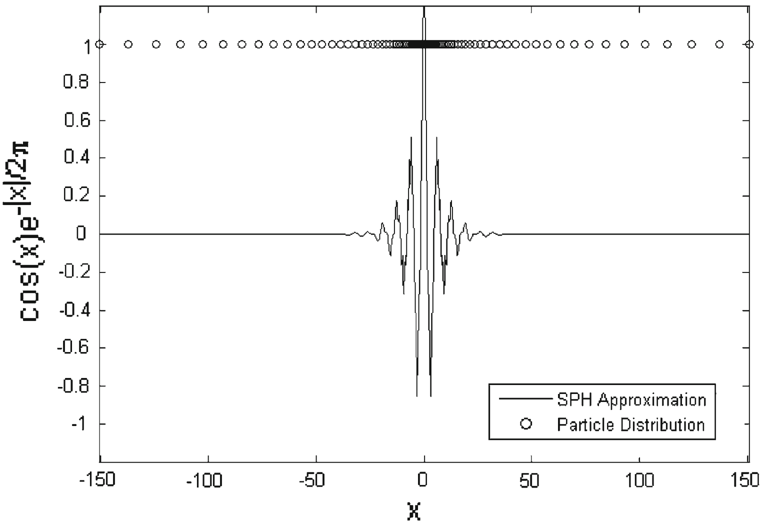


**Fig. 3** The approximation obtained without any consistency restoring for the function $f(x) = \cos(x)\, e^{-|x|/2\pi}$ with the S-LIU particle distribution

## 4 Conclusions and future works

In this paper the numerical behaviors of SPH with respect to the consistency restoring problem and to the particle distribution choice have been presented and useful hints on how these two aspects affect the goodness of the approximation have been proposed. Various numerical tests approximating 1D, 2D and 3D functions demonstrate the effectiveness of the introduced idea.

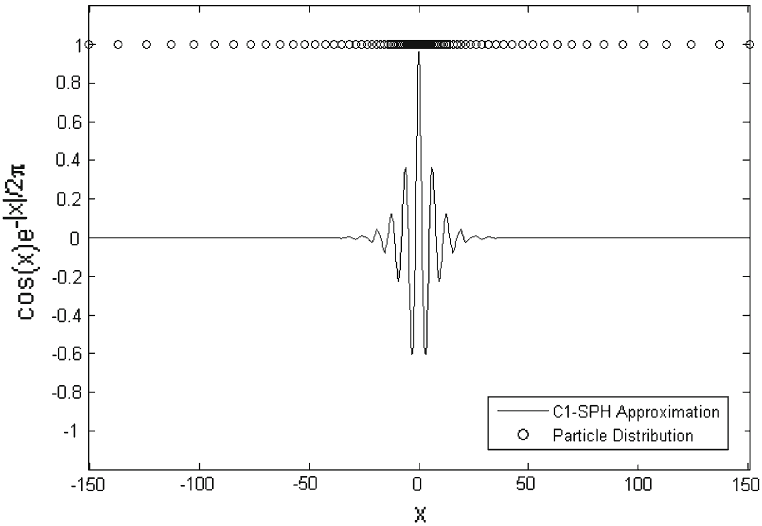


**Fig. 4** The approximation obtained by restoring the 1-st order of consistency for the function $f(x) = \cos(x)\, e^{-|x|/2\pi}$ with the S-LIU particle distribution

Future works will be devoted to investigate the usage of the SPH framework in the Image Processing context.

## References


1. G. Ala, E. Francomano, A. Tortorici, E. Toscano, F. Viola, A smoothed particle interpolation scheme for transient electromagnetic simulation. IEEE Trans. Magn. **42**, 647–650 (2006)
2. G. Ala, E. Francomano, A. Tortorici, E. Toscano, F. Viola, Corrective meshless particle formulations for time domain Maxwell's equations. J. Comput. Appl. Math. **210**, 34–46 (2007)
3. T. Belytschko, Y. Krongauz, J. Dolbow, C. Gerlach, On the completeness of meshfree methods. Int. J. Numer. Methods Eng. **43**, 785–819 (1998)
4. T. Belytschko, Y. Krongauz, D. Organ, M. Fleming, P. Krysl, Meshless methods: an overview and recent developments. Comput. Meth. Appl. Mech. Eng. **139**, 3–47 (1996)
5. J. Bonet, T.S.L. Lok, Variational and momentum preservation aspects of smooth particle hydrodynamics formulations. Comput. Meth. Appl. Mech. Eng. **180**, 97–115 (1999)
6. J. Bonet, S. Kulasegaram, A simplified approach to enhance the performance of smooth particle hydrodynamics methods. Appl. Math. Comput. **126**, 133–155 (2002)
7. J.K. Chen, J.E. Beraun, T.C. Carney, A corrective smoothed particle method for boundary value problems in heat conduction. Int. J. Numer. Methods Eng. **46**, 231–252 (1999)
8. G. Di Blasi, E. Francomano, A. Tortorici, E. Toscano, On the Consistency Restoring in SPH. in *Proceedings of the CMMSE (International Conference on Computational and Mathematical Methods in Science and Engineering)* (2009)
9. D.A. Fulk, D.W. Quinn, An analysis of 1-D smoothed particle hydrodynamics kernels. J. Comput. Phys. **126**, 165–180 (1996)
10. R.A. Gingold, J.J. Monaghan, Smoothed particle hydrodynamics: theory and application to nonspherical stars. Mon. Not. Roy. Astron. Soc. **181**, 375–389 (1985)
11. L. Hernquist, N. Katz, TreeSPH: a unification of SPH with the hierarchical tree method. Astrophys. J. Suppl. Ser. **70**, 419–446 (1989)
12. P. Laguna, Smoothed particle interpolation. Astrophys. J. **439**, 814–821 (1994)

13. M. Lastiwka, N. Quinlan, M. Basa, Adaptive particle distribution for smoothed particle hydrodynamics. Int. J. Numer. Methods Fluids **47**, 1403–1409 (2005)
14. G.R. Liu, M.B. Liu, *Smoothed Particle Hydrodynamics—a Mesh-Free Particle Method* (World Scientific Publishing, Singapore, 2003)
15. G.R. Liu, *Mesh Free Methods—Moving beyond the Finite Element Method* (CRC Press, Boca Raton, 2003)
16. M.B. Liu, G.R. Liu, Restoring particle consistency in smoothed particle hydrodynamics. Appl. Numer. Math. **56**, 19–36 (2006)
17. M.B. Liu, G.R. Liu, K.Y. Lam, Constructing smoothing functions in smoothed particle hydrodynamics with applications. J. Comput. Appl. Math. **155**, 263–284 (2003)
18. J.J. Monaghan, An introduction to SPH. Comput. Phys. Commun. **48**, 89–96 (1988)
19. J.J. Monaghan, Smoothed particle hydrodynamics. Annu. Rev. Astron. Astrophys. **30**, 543–574 (1992)
20. J.J. Monaghan, J.C. Lattanzio, A refined particle method for astrophysical problems. Astron. Astrophys. **149**, 135–143 (1985)
21. J.P. Morris, Analysis of Smoothed Particle Hydrodynamics with Applications. Ph.D. Thesis, Monash University, (1996)
22. P.R. Shapiro, H. Martel, J.V. Villumsen, J.M. Owen, Adaptive smoothed particle hydrodynamics, with application to cosmology: methodology. Astrophys. J. Suppl. Ser. **103**, 269–330 (1996)
23. D. Shepard. A Two Dimensional Function for Irregularly Spaced Data. in Proceedings of the ACM National Conference, (1968)